Omega-syntactic congruences*

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Abstract

An ω-language over a finite alphabet X is a set of infinite sequences of letters of X. Previously studied syntactic equivalence relations defined by ω-languages have mainly been relations on X^*. In this paper the emphasis is put on relations in X^ω, by associating to an ω-language L a congruence on X^ω, called the ω-syntactic congruence of L. Properties of this congruence and notions induced by it, such as ω-residue, ω-density, and separativeness are defined and investigated. Finally, a congruence on X^* related to the ω-syntactic congruence and quasi-orders on X^ω induced by an ω-language are studied.

Keywords: ω-syntactic congruence, ω-language, dense language, disjunctive language, residue, syntactic monoid.

1 Introduction

Various types of congruences on X^* have been introduced in connection with ω-words and ω-languages. The usual equivalence relations induced by an ω-language L on X^* are R_L and P_L, defined by (see, for example, [6]):

$$w \equiv v(R_L) \Leftrightarrow (\forall y \in X^\omega, wy \in L \text{ iff } vy \in L)$$

$$w \equiv v(P_L) \Leftrightarrow (\forall x \in X^*, y \in X^\omega, xwy \in L \text{ iff } xvy \in L).$$

Both R_L and P_L are equivalence relations on X^* which coincide with the Nerode and syntactic equivalence if L is a language over X^*. One easily proves that R_L is a right congruence and that P_L is a congruence. The monoid $\mathrm{Syn}(L) = X^*/P_L$ is called the *syntactic monoid* of L.

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An ω-language L is said to be disjunctive or right disjunctive if the corresponding relation P_L or R_L is the equality. It is dense if for every $u \in X^*$ there exist $x \in X^*$, $y \in X^\omega$ such that $xuy \in L$. Obviously, if an ω-language is disjunctive, it is dense. If the index of P_L is finite, then L is said to be μ-regular. (μ-regular ω-languages are sometimes referred to as finite-state ω-languages; see, for example, [7]) Remark that the index of P_L is finite if and only if the index of R_L is finite.

Disjunctive and right-disjunctive ω-languages and their properties have been studied in [4]. Syntactic monoids of ω-languages and conditions under which they are trivial have been investigated in [5]. In [6] it is shown that every finitely generated monoid is isomorphic with the syntactic monoid of an ω-language.

The congruences that have been previously defined for ω-languages are mainly congruences on X^* (see [1], [7], [9], [10]) and consequently all the notions related to these congruences mainly refer to the set X^*. However, it is possible to define congruences on X^ω, in particular a congruence on X^ω induced by an ω-language.

The ω-syntactic congruence associated with an ω-language $L \subseteq X^\omega$ will be denoted by S_L. Connected with the ω-syntactic congruence S_L, one can define the notions of ω-residue, ω-density and separativeness, which are the counterparts in X^ω of the classical notions of residue, density and disjunctivity. The equivalence of the finitness of R_L to the finiteness of S_L implies that μ-regularity is also characterized by the finitness of S_L.

This paper studies the ω-syntactic congruence, its properties and related topics. Moreover, a *quasi-order* on X^ω is introduced and its relations with separative ω-languages and other related notions are investigated.

2 Omega-syntactic congruences

An *alphabet* X is a finite nonempty set. X^* is the free monoid generated by it under the catenation operation. The elements of X^* are words; in particular, 1 is the *empty word*, and $X^+ = X^* \backslash \{1\}$. X^ω is the set of ω-*words*, that is, of infinite sequences over X. The length of a word $w \in X^*$ will be denoted by $|w|$ and the cardinality of a set X by $\text{card}(X)$. The catenation of two words u, v will be denoted either by uv or by $u.v$.

Let M be a monoid with identity 1. An *operand over* M (see, for example [3]) is a nonempty set T such that:

– with every pair $x \in M, u \in T$ is associated an element $xu \in T$ called the product of x and u;

– $(xy)u = x(yu)\ \forall x, y \in M, u \in T$;

– $1.u = u\ \forall u \in T$.

A nonempty subset T' of T such that $u \in T'$ implies $xu \in T'$, $\forall x \in M$, is called a suboperand of T over M and T' itself is an operand over M.

For example, if X^* and X^ω are respectively the set of words and the set of ω-words over X, then X^ω is an operand over X^*.

An equivalence relation ρ over X^ω is said to be *compatible* if

$$r \equiv s\ (\rho)\ \ \Rightarrow\ \ xr \equiv xs\ (\rho)\ \ \forall x \in X^*$$

A compatible equivalence relation will also be called simply a *congruence.*

Remark. Let ρ be a congruence over X^ω and let $T = \{[u] \mid u \in X^\omega\}$ be the set of all the classes of ρ ($[u]$ denotes the class containing u). Define the product of $x \in X^*$ and $[u]$ by $x[u] = [xu]$. Since ρ is a congruence, it is easy to see that this product is well defined (i.e. it does not depend on the choice of representatives for a given class). It follows then that T is an operand over X^*, called the quotient-operand modulo ρ.

Definition 2.1 *An ω-language L defines on X^ω a binary relation S_L by:*

$$r \equiv s\ (S_L)\ \ \textit{iff}\ \ (xr \in L\ \Leftrightarrow xs \in L), x \in X^*, r, s \in X^\omega$$

i.e. $Lr^{-1} = Ls^{-1}$*, where* $Lr^{-1} = \{x \in X^* \mid xr \in L\}$*.*

The relation S_L is a congruence, i.e., a compatible equivalence relation, and will be called in the sequel the *ω-syntactic congruence* of L (see [10] for a similar notion).

An ω-language L is called (see [5])

– a *left ω-ideal* if $X^*L \subseteq L$ (i.e. if L is an X^*-subset);

– *suffix closed* or simply *suf-closed* if $X^{*[-1]}L \subseteq L$, i.e., if $xu \in L$ implies $u \in L$.

– *absolutely closed* if $L = X^*L'$ for a suf-closed ω-language L'.

For example, the ω-language $L = X^*a^\omega$ over $X = \{a, b\}$ is a left ω-ideal and it is suffix closed. The ω-language a^ω is suf-closed and hence L is absolutely closed. Remark that an absolutely closed ω-language is always a left ω-ideal.

The ω-language $W(L) = \{u \in X^\omega | Lu^{-1} = \emptyset\}$ is called the ω $-$ *residue* of $L \subseteq X^\omega$.

Proposition 2.1 *Let L be an ω-language. The ω-syntactic congruence of L has the following properties:*

(i) L is a union of classes of S_L;

(ii) If R is a congruence and if L is a union of classes of R, then $R \subseteq S_L$.

(iii) If nonempty, the ω-residue $W(L)$ is a class of S_L and a left ω-ideal.

Proof. *(i)* Let $u \in L$ and suppose that $u \equiv v\ (S_L)$. Since $1 \in Lu^{-1} = Lv^{-1}$, the word $v = 1.v$ belongs to L.

(ii) Suppose that $u \equiv v\ (R)$. If $x \in Lu^{-1}$, then $xu \in L$. From the compatibility of R it follows that $xu \equiv xv\ (R)$. The facts that L is a union of classes of R and $xu \in L$, imply that $xv \in L$, $x \in Lv^{-1}$. Consequently, $Lu^{-1} \subseteq Lv^{-1}$. Similarly one can prove that $Lv^{-1} \subseteq Lu^{-1}$. Therefore, $u \equiv v\ (S_L)$ which implies $R \subseteq S_L$.

(iii) Immediate because $u \in W(L)$ if and only if $Lu^{-1} = \emptyset$. $\square$

Corollary 2.1 *If T is a class of a congruence R over X^ω, then $R \subseteq S_T$.*

Proof. This is a special case of (ii). □

Given an ω-language L, the *index* of R_L (respectively S_L) is the cardinality of the set of classes of R_L (respectively S_L).

Recall that an ω-language L is called *μ-regular* if the index of R_L is finite. The next result shows that the μ-regularity of an ω-language can be characterized either by the finiteness of the index of R_L in X^* or by the finiteness of S_L in X^ω.

Proposition 2.2 *An ω-language $L \subseteq X^\omega$ is μ-regular if and only if the index of S_L is finite.*

Proof. If the ω-language $L \subseteq X^\omega$ is μ-regular then the index of R_L is finite and therefore the set $\{w^{-1}L| \quad w \in X^*\}$ is finite. Remark that $Lu^{-1} = \bigcup_{w \in Lu^{-1}} [w]_{R_L}$. Indeed, if $x \in [w]_{R_L}$ for some $w \in Lu^{-1}$ then $x \equiv w \ (R_L)$ and $wu \in L$. This implies that for all $v \in X^\omega$, $xv \in L$ iff $wv \in L$. In particular, $wu \in L$ implies $xu \in L$, that is, $x \in Lu^{-1}$. The other inclusion is obvious. If R_L is of finite index, the union is finite and there are only finitely many different unions, therefore the index of S_L is finite.

Conversely, note that $w^{-1}L = \bigcup_{u \in w^{-1}L} [u]_{S_L}$. Indeed if $v \in [u]_{S_L}$, $u \in w^{-1}L$ then $v \equiv u \ (S_L)$ and $wu \in L$. As $wu \in L$ iff $wv \in L$ we have $v \in w^{-1}L$. The other inclusion is obvious. If S_L is of finite index then the union is finite and there are only finitely many different unions. This further implies that the index of R_L is finite, i.e., L is μ-regular. □

Example 1 X^ω is μ-regular because the index of S_L is 1.

Example 2 $L = a^\omega = aaa \cdots aaa \cdots$ over $X = \{a, b\}$ is μ-regular. The classes of S_L are a^ω and the ω-residue $W(a^\omega)$. Therefore the index of S_L is 2.

Example 3 Let $L = \{a^n b a^\omega | n \geq 1\}$ over $X = \{a, b\}$. The classes of S_L are L, ba^ω, a^ω and $W(L)$ and the index of S_L is then 4.

If X^* is ordered lexicographically, $X^* = \{a, b, a^2, ab, ba, b^2, \cdots\}$, then the disjunctive ω-word $u = aba^2abbab^2 \cdots$ is not μ-regular because $S_{\{u\}}$ has an infinite index.

Proposition 2.3 *Let L, L_1, L_2 be ω-languages in X^ω. Then:*

(i) $S_L = S_{\bar{L}}$ where $\bar{L}$ denotes the complement of L in X^ω;

(ii) $S_{L_1} \cap S_{L_2} \subseteq S_{L_1 \cup L_2}$;

(iii) $S_{L_1} \cap S_{L_2} \subseteq S_{L_1 \cap L_2}$.

(iv) If $T \subseteq X^*$ and $L, T \neq \emptyset$, then $S_L \subseteq S_{T^{-1}L}$ where $T^{-1}L = \{u \in X^\omega \mid \exists t \in T, tu \in L\}$.

Proof. *(i)* Immediate.

(ii) Let $u \equiv v\ (S_{L_1} \cap S_{L_2})$, that is, $L_1u^{-1} = L_1v^{-1}$ and $L_2u^{-1} = L_2v^{-1}$. If $xu \in L_1 \cup L_2$, then $xu \in L_1$ or $xu \in L_2$, hence $xv \in L_1$ or $xv \in L_2$. Therefore $xv \in L_1 \cup L_2$, that is, $(L_1 \cup L_2)u^{-1} \subseteq (L_1 \cup L_2)v^{-1}$. By symmetry $(L_1 \cup L_2)v^{-1} \subseteq (L_1 \cup L_2)u^{-1}$ which implies $u \equiv v\ (S_{L_1 \cup L_2})$.

(iii) By *(i)* and *(ii)*, we have:

$$S_{L_1} \cap S_{L_2} = S_{\bar{L}_1} \cap S_{\bar{L}_2} \subseteq S_{\bar{L}_1 \cup \bar{L}_2} = S_{L_1 \cap L_2}.$$

(iv) Suppose $u \equiv v\ (S_L)$, that is, $Lu^{-1} = Lv^{-1}$. If $x \in T^{-1}Lu^{-1}$, then $xu \in T^{-1}L$ and $txu \in L$ for some $t \in T$. Hence $tx \in Lu^{-1} = Lv^{-1}$, $txv \in L$ and $xv \in T^{-1}L$. Therefore $x \in T^{-1}Lv^{-1}$ which shows that $T^{-1}Lu^{-1} \subseteq T^{-1}Lv^{-1}$. By symmetry, the converse inclusion also holds. Hence $u \equiv v\ (S_{T^{-1}L})$. □

The following proposition shows that all the congruences over X^ω can be obtained from the ω-syntactic congruences.

Proposition 2.4 *Every congruence R (over X^ω) is the intersection of ω-syntactic congruences. More precisely, there exists a family of ω-languages $\Phi(R) = \{L_i | i \in I\}$ such that:*

$$R = \bigcap_{i \in I} S_{L_i}$$

Proof. We can choose, for example, the family $\Phi(R)$ to be the family of all the classes L_i of R. By Corollary 2.1, if L_i is a class of R, then $R \subseteq S_{L_i}$, hence $R \subseteq \bigcap_{i \in I} S_{L_i}$.

Suppose now that $u \equiv v\ (\bigcap_{i \in I} S_{L_i})$ and let L_j be the class of R containing u. Then $u \equiv v\ (S_{L_j})$, that is, $L_ju^{-1} = L_jv^{-1}$. As $1.u = u \in L_j$, we have $1 \in L_ju^{-1} = L_jv^{-1}$ which implies $1.v = v \in L_j$. Because L_j is a class of R, it follows that $u \equiv v\ (R)$ therefore $\bigcap_{i \in I} S_{L_i} \subseteq R$. Consequently,

$$R = \bigcap_{i \in I} S_{L_i}. \quad \square$$

Recall that (see [5]) an ω-language $L \subseteq X^\omega$ is absolutely closed if and only if the syntactic monoid of L, $\mathrm{Syn}(L)$ is trivial, that is, $\mathrm{card}(\mathrm{Syn}(L)) = 1$. This is equivalent to the fact that P_L is the universal relation, i.e. has a unique class. (Here $\mathrm{Syn}(L) = X^*/P_L$.)

Proposition 2.5 *Let L be an absolutely closed ω-language, $L \neq X^\omega$. Then S_L has only two classes, L and the ω-residue $W(L)$.*

Proof. By a result of [5], the ω-language L is absolutely closed if and only if L and the complement $\bar{L}$ of L are left ω-ideals.

Let $u \in L$. Then $Lu^{-1} = X^*$ and L is contained in a class of S_L. Since L is a union of classes of S_L, it follows that L is a class of S_L. The complement $\bar{L}$ of L being a left ω-ideal is therefore contained in the ω-residue $W(L)$. Since L and $W(L)$ are classes of S_L, they are the only two classes of S_L. □

Proposition 2.5 does not hold anymore in case L is a suf-closed ω-language. For example, let $X = \{a, b\}$ and $L = \{a^\omega, b^\omega\}$. L is suf-closed but S_L has three classes: a^ω, b^ω and $W(L)$.

In fact, there exist suf-closed ω-languages with the property that S_L has infinitely many classes.

Indeed, let $X = \{0, 1, 2, \cdots, 9\}$ and let

$$\begin{array}{rcl} u_1 & = & 12345678910111213\cdots \\ u_2 & = & 234567891011121314\cdots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots \\ u_n & = & n(n+1)(n+2)\cdots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Then L is suf-closed but S_L has infinitely many equivalence classes.

A nonempty ω-language L is called *suffix-free* (*outfix-free*) or simply *suf-free* *(out-free)* if $u, xu \in L$ *(yu, yxu* $\in L)$ implies $x = 1$. An out-free ω-language is always suf-free.

For example, the ω-language $L = aba^2b^2\cdots a^nb^n\cdots$ over $X = \{a, b\}$ is out-free.

Proposition 2.6 *Let L be an ω-language. Then:*

(i) If L is suf-free, L is a class of S_L.

(ii) If L is out-free, then every class T of S_L, $T \neq W(L)$, is a suf-free ω-language.

Proof. *(i)* If $u \in L$, then $Lu^{-1} = \{1\}$, hence L is contained in a class T of S_L . If $v \in T$, then $Lv^{-1} = \{1\}$ and therefore $v = 1.v \in L$. Consequently, $L = T$.

(ii) Suppose that u and $xu = v$ are words in T. Since $T \neq W(L)$, there exists $y \in X^*$ such that $yu \in L$. From $u \equiv v\ (S_L)$ it follows that $yu \equiv yv\ (S_L)$. Since $yu \in L$, $yv = yxu \in L$. On the other hand, the fact that L is out-free implies that $x = 1$, that is T is suf-free. □

3 Omega-dense and separative ω-languages

An ω-language $L \subseteq X^\omega$ is called *dense* iff for any word $x \in X^*$, there exist $u \in X^*$ and $y \in X^\omega$ such that $uxy \in L$. In other words, L is called dense iff any word of X^* occurs as a subword of a word of L.

One can generalize the notion of density from X^* to X^ω in the following natural fashion. An ω-language L will be called ω-dense if any infinite word occurs as a subword of a word in L. Formally,

Definition 3.1 *An ω-language L is called ω-dense if its ω-residue $W(L) = \emptyset$.*

If $Suf(L) = \{v \in X^\omega | \exists x \in X^*, \exists u \in L, u = xv\}$ is the set of all suffixes of words in L, then it is immediate that L is ω-dense iff $Suf(L) = X^\omega$.

Let $X = \{a, b, \cdots\}$. Then X^ω, aX^ω and bX^ω are examples of ω-dense ω-languages. Generally if L is ω-dense, then AL is ω-dense for all $A \subseteq X^*, A \neq \emptyset$.

Remark that, if $\{x_1, x_2, \cdots\}$ is any ordering of X^+, the ω-word $u = x_1 x_2 \ldots$ obtained by catenating the ordered sequence $\{x_1, x_2, \ldots\}$ is disjunctive, hence dense. However, the following proposition shows that u is not ω-dense.

Proposition 3.1 *Every ω-dense ω-language L over an alphabet X with at least two letters, is infinite.*

Proof. Let $X = \{a, b, \cdots\}$ be an alphabet of cardinality greater than 1 and assume, by reductio ad absurdum, that the ω-dense ω-language $L = \{u_1, u_2, \cdots, u_n\}$ over X is finite.

Consider the finite language $\{v_i | \ 1 \leq i \leq n+1\}$ where $v_i = (a^i b^i)^\omega$. Since L is ω-dense, there exist words $x_1, x_2, \cdots, x_n, x_{n+1}$ such that

$$x_1 v_1 \in L, x_2 v_2 \in L, \cdots, x_n v_n \in L, x_{n+1} v_{n+1} \in L.$$

As L contains only n distinct words, the equality $u_i = x_i v_i = x_j v_j$ will hold for some $i \neq j$, that is,

$$u_i = x_i (a^i b^i)^\omega = x_j (a^j b^j)^\omega, i \neq j.$$

This implies $a = b$ − a contradiction. Consequently, our assumption that L is finite was false.□

Recall (see, for example, [8]) that a subset P of X^* is called *dense* (in X^*) if for every $w \in X^*$ there exist words $x, y \in X^*$ such that xwy belongs to P.

For $L \subseteq X^\omega$, let $Prf(L) = \{x \in X^* | \exists u \in L, u = xv\}$ be the set of all prefixes of the words in L. The next result gives a connection between the ω-density of an ω-language and the density of the set if its prefixes.

Proposition 3.2 *If L is ω-dense, then $Prf(L)$ is dense (in X^*).*

Proof. Let w be a word in X^* and consider the ω-word w^ω. As L is an ω-dense ω-language, there exists $x \in X^*$ such that $xw^\omega \in L$. This implies, for example, that xww is in Prf(L). This means that we have found the words $x, w \in X^*$ with the property $xww \in$Prf(L), which assures that Prf(L) is dense. □

The notion of density in X^* is closely connected with the notion of disjunctivity. An ω-language is disjunctive if its congruence P_L is the equality. A disjunctive ω-language is dense, but the converse does not hold. An analogous of disjunctivity when considering relations over X^ω is the separativeness.

An ω-language L is *separative* if its ω-syntactic congruence separates all the words of X^ω: every word of X^ω belongs to a different equivalence class.

Definition 3.2 *An ω-language L is called*
(i) separative iff $Lu^{-1} = Lv^{-1}$ implies $u = v$.
(ii) quasi-separative iff $Lu^{-1} = Lv^{-1} \neq \emptyset$ implies $u = v$.

In other words, L is separative if S_L is the identity and quasi-separative if S_L is the identity outside the ω-residue $W(L)$.

It is easy to see that L is separative iff for every pair $u, v \in X^\omega$, $u \neq v$, there exists $x \in X^*$ such that $xu \in L$, $xv \notin L$ or vice versa. Simple examples seem to be difficult to find. The following proposition shows how to obtain separative languages from special types of partitions of Y^+, where $Y \subset X$ and $|X| \geq 2$.

Remark that if $|X| = 1$ then $|X^\omega| = 1$ and the language X^ω is trivially separative.

Proposition 3.3 *Let X be a finite alphabet with $|X| \geq 2$, let $a \in X$ and let $Y = X \setminus \{a\}$. Furthermore let $\Pi = \{Y_0, Y_1, \ldots, Y_n, \ldots\}$ be a partition of Y^+ with infinitely many classes, all of them infinite. Then there exists a separative language asociated to this partition.*

Proof. For $n \geq 0$, let

$$T_n = \{u \in X^\omega | u = a^n w, w \notin aX^\omega\}.$$

Let $\mathbf{c}$ be the cardinality of the set of the real numbers. The set X^ω and the sets $T_n, n \geq 0$, have the same cardinality $\mathbf{c}$. Consequently, these sets can be listed using the same set I of indices where I has the cardinality $\mathbf{c}$:

$$T_n = \{u_{n_i} \in X^\omega | i \in I\}, n \geq 0.$$

Each class Y_n of the partition Π contains infinitely many words of Y^+. For each n, let P_n be the set of all nonempty subsets of Y_n. Clearly the cardinality of each P_n is $\mathbf{c}$. This implies in particular that the elements of P_n can be listed using the same index set I:

$$P_n = \{S_{n_i} | i \in I, S_{n_i} \subseteq Y_n, S_{n_i} \neq \emptyset\}$$

Furthermore, $S_{n_i} \cap S_{m_j} = \emptyset$ for $n \neq m$.

The ω-language L is defined in the following way.

For each $n \geq 0$ and $i \in I$, let $L_{n_i} = S_{n_i} a u_{n_i}$. Then:

$$L = \bigcup_{n \geq 0, i \in I} L_{n_i} \cup a^\omega$$

Let us show that L is separative, that is, $Lu^{-1} \neq Lv^{-1}$ for all $u, v \in X^\omega$, $u \neq v$.. We have to consider the following cases.

Case 1. $u \in T_m$, $v \in T_n$ with $m \neq n$. Then $u = u_{m_i}$, $v = u_{n_j}$ with $i, j \in I$ and $u = a^m b\alpha$, $v = a^n c\ \beta$ where $m, n \geq 0$, $b, c \in Y$ and $\alpha, \beta \in X^\omega$.

Without loss of generality, we can assume that $m < n$. If $x \in S_{m_i}$, then $xau = xau_{m_i} \in L_{m_i} \subseteq L$. By the definition of the sets S_{r_k}, $x \in S_{m_i}$ implies that $x \notin S_{r_k}$ for $r \neq m$. Therefore $xav = xau_{n_j} \notin L$ and $Lu^{-1} \neq Lv^{-1}$.

Case 2. $u \in T_n, v \in T_n$ and $u = u_{n_i}$, $v = u_{n_j}$. Since $u \neq v$, we must have $i \neq j$. Furthermore $u = a^n b\alpha$, $v = a^n c\beta$, $b, c \in Y$, $\alpha, \beta \in X^\omega$. Since $i \neq j$, then we have $S_{n_i} \neq S_{n_j}$. Hence there exists $x \in S_{n_i}$, $x \notin S_{n_j}$ or vice versa. Suppose the first case. Then $xau \in L_{n_i} \subseteq L$, but $xav \notin L_{n_j}$. Since $v = u_{n_j}$, then, from the definition of the ω-languages L_{r_s} and L, we have $xau_{n_j} \in L$ iff $xau_{n_j} \in L_{n_j}$ iff $x \in S_{n_j}$, in contradiction with $x \notin S_{n_j}$.

Case 3. $u \in T_n$, $v = a^\omega$. Suppose $Lu^{-1} = Lv^{-1}$. Clearly $a^k \in Lv^{-1}$ for $k \geq 0$. Since $u = u_{n_i} = a^n b\alpha$, then $a^k aa^n b\alpha \in L$ for $k \geq 0$, a contradiction because $xaa^n b\alpha \in L$ implies $x \in Y^+$ and $a^k \notin Y^+$.□

Proposition 3.4 *Every separative ω-language L is ω-dense.*

Proof. If L is not ω-dense, then its ω-residue W(L) is non empty. Furthermore, W(L) is infinite and W(L) is a class of the congruence S_L. Since L is separative, S_L is the identity, a contradiction. □

While it is difficult to find simple examples of separative ω-languages, this is no more the case for quasi-separative ω-languages as shown in the following proposition.

Proposition 3.5 *Every ω-word u is quasi-separative.*

Proof. Let u be an ω-word and let $r \equiv s$ (S_u) with $r, s \notin W(u)$. Then there exist $x, y \in X^*$ such that $u = xr = ys$. The equality $ur^{-1} = us^{-1}$ implies $x \in us^{-1}$ and $u = xr = ys = xs$. This further implies $r = s$, therefore u is quasi-separative. □

Let $L \subseteq X^\omega$ be an ω-language with a non empty ω-residue $W(L)$. We can construct a congruence ρ on X^ω in the following way: $W(L)$ is a class ρ and all the other classes of ρ are the singletons taken from the complement of $W(L)$. By analogy with semigroups, such a congruence will be called the *Rees congruence* associated with the ω-residue of L (see [3]).

Proposition 3.6 *Let L be an ω-language such that $W(L) \neq \emptyset$. Then*

(i) L is quasi-separative $\Leftrightarrow$ S_L is the Rees congruence associated with $W(L)$.

(ii) L is quasi-separative and $\bar{L} = W(L)$ $\Leftrightarrow$ S_L is the identity on L and $\bar{L} = X^\omega \backslash L$ is a class of S_L.

Proof. *(i)* "$\Rightarrow$" Since $u \in W(L)$ if and only if $Lu^{-1} = \emptyset$, it is clear that $W(L)$ is a class of S_L. Since S_L is the identity on $X^\omega \backslash W(L)$, it follows that S_L is the Rees congruence associated with $W(L)$.

"$\Leftarrow$" Since the Rees congruence is the identity outside the ω-residue, S_L is the identity on $X^\omega \backslash W(L)$, that is, L is quasi-separative.

(ii) "$\Rightarrow$" If $u, v \in L$, we have $Lu^{-1} = Lv^{-1} = \emptyset$, that is, $u \equiv v(S_L)$. If $u, v \in L$, $u \neq v$, then $Lu^{-1} \neq \emptyset$, $Lv^{-1} \neq \emptyset$. Since L is quasi-separative, we have $Lu^{-1} \neq Lv^{-1}$, i.e., u is not equivalent to v modulo S_L. Therefore S_L is the identity on L and $\bar{L}$ is a class of S_L.

"$\Leftarrow$" Suppose that $Lu^{-1} = Lv^{-1} \neq \emptyset$. This implies there exists $x \in X^*$ such that $xu, xv \in L$. Since $u \equiv v(S_L)$ and S_L is compatible, $xu \equiv xv(S_L)$. Since S_L is the identity on L and $xu, xv \in L$, then $xu = xv$ therefore $u = v$. This means L is separative. It is easy to see that $W(L) = \bar{L}$. □

Proposition 3.7 *Let L be a μ-regular ω-language. If L is quasi-separative, then L is finite.*

Proof. The ω-language L is a union of classes of S_L (Proposition 2.1) Since L is μ-regular, the index of S_L is finite(Proposition 2.2). If L is not finite, then there exists a class T of S_L such that $T \subseteq L$ and T infinite. If $u, v \in T$ with $u \neq v$, then $Lu^{-1} = Lv^{-1} \neq \emptyset$, a contradiction because L is separative. Therefore L is finite. □

4 Congruences S_L and P_L

In this section, we consider a connection between the ω-syntactic congruence S_L on X^ω and the congruence P_L on X^* associated with an ω-language L.

With every congruence ρ on X^ω, one can associate a congruence $s(\rho)$ on X^* defined in the following way:

$$c\ s(\rho)\ d \ \Leftrightarrow\ cu \equiv du\ (\rho)\ \forall u \in X^\omega.$$

Proposition 4.1 *The relation $s(\rho)$ is a congruence on X^*.*

Proof. It is immediate that $s(\rho)$ is an equivalence relation. Since ρ is compatible, it follows then that $s(\rho)$ is left compatible. Let $x \in X^*$. Since $xu \in X^\omega$, from $cu \equiv du\ (\rho)$ for all u, it follows that $cxu \equiv dxu\ (\rho)$ (take for u the word xu). Hence $cxu \equiv dxu\ (\rho)$ for all $u \in X^\omega$ therefore $cx \equiv dx\ (s(\rho))$, which implies that $s(\rho)$ is right compatible. Consequently, $s(\rho)$ is a congruence of X^*. □

Remark. If ρ is the universal relation on X^ω, then $s(\rho)$ is also the universal relation on X^*. If ρ is the identity on X^ω, then $s(\rho)$ is the identity on X^*.

The next proposition shows how the congruence P_L of L is related to the ω-syntactic congruence S_L.

Proposition 4.2 *If $L \subseteq X^\omega$, then $P_L = s(S_L)$.*

Proof. Suppose that $c \equiv d\ (P_L)$, $c, d \in X^*$. This means that, for every $x \in X^*$ and $u \in X^\omega$, $xcu \in L \Leftrightarrow xdu \in L$. This further implies $cu \equiv du\ (S_L)$ for every $u \in X^\omega$ and hence $c\ s(S_L)\ d$. Therefore $P_L \subseteq s(S_L)$.

Suppose now that $c\ s(S_L)\ d$, i.e. $cu \equiv du\ (S_L)\ \forall u \in X^\omega$. This implies that for every $x \in X^*$, $xcu \in L \Leftrightarrow xdu \in L$, i.e. $c \equiv d\ (P_L)$, and $s(S_L) \subseteq P_L$.

Therefore $P_L = s(S_L)$. □

We give now a few examples of the connection between S_L and P_L.

Example 1 Let $L = a^\omega$ over $X = \{a, b\}$. The classes of S_L are a^ω and $W(a^\omega)$. The classes of P_L are a^* and $\{xby | x, y \in X^*\}$. The syntactic monoid of L is isomorphic to the monoid consisting of only 1 and 0.

Example 2 Let $L = \{a^\omega, b^\omega\}$ over $X = \{a, b\}$. Then the classes of S_L are a^ω, b^ω and $W(L)$. The classes of P_L are $\{1\}$, a^+, b^+ and $X^+ \backslash \{a^+, b^+\}$.

Example 3 If $L = \{a^n ba^\omega | n \geq 1\}$ over $X = \{a, b\}$, then the classes of S_L are L, a^ω, ba^ω, $W(L)$. The classes of P_L are $\{1\}$, a^+, $\{b\}$, $\{ab\}$, $bX^+ \cup X^* b^2 X^*$.

Example 4 Let $L = \{u\}$ be an ω-word. By Proposition 3.5, u is quasi-separative and the classes of S_u are the ω-residue $W(L)$ (if not empty) and the singletons consisting of the ω-words in $U = X^\omega \backslash W(L)$. Let

$$u = a_1 a_2 \ldots a_k \ldots, \quad u_1 = u, u_2 = a_2 \ldots a_k \ldots, \quad u_k = a_k a_{k+1} \ldots$$

$$v_0 = a_1, \quad v_1 = a_1 a_2, \ldots \quad , v_k = a_1 a_2 \ldots a_{k+1}, \ldots$$

where a_i are letters in X. Then $U = \{u_1, u_2, \ldots, u_k, \ldots\}$.

Let $T = \{v_k | \quad k \geq 0\}$. If $x \equiv v_k(P_L)$, then, in particular, $xu_k \equiv v_k u_k(S_L)$. Since $v_k u_k = u \in L$, we have $xu_k \equiv u(S_L)$, i.e., $rxu_k = u$ iff $ru = u$. Hence $r = 1$, $xu_k = u = v_k u_k$ and $x = v_k$. Therefore every word in T is a class of P_L.

Let $x \notin T$. If $xw \in T$, then $xw = u_i$ with $u = v_i u_i$ and $v_i xw = v_i u_i = u$. Since this is true for u in particular, $v_i xu = u$, $x = 1$ and $x \in T$ – a contradiction. It follows then that $xw \in W(L)$, a class of S_L. Hence $x, y \in T$ implies $xw, yw \in W(L)$, i.e., $xw \equiv yw(S_L)$ for all $w \in X^\omega$ and $x \equiv y(P_L)$. Therefore $\bar{T} = X^* \backslash T$ is a class of P_L.

5 Compatible quasi-orders and orders on X^ω

Recall that a binary relation σ on a set S is called a *quasi-order* if it is reflexive and transitive (see, for example [2]).

If L is an ω-language, the relation $\sigma(L)$ defined by

$$u\ \sigma(L)\ v \ \Leftrightarrow \ Lu^{-1} \subseteq Lv^{-1}$$

is a *quasi-order* on X^ω that will be called the *principal quasi-order* associated with the ω-language L.

If σ is a quasi-order on X^ω, then σ is said to be *compatible* if $u, v \in X^\omega$, $x \in X^*$ and $u \sigma v$ imply $xu \sigma xv$.

In this section we show that all the compatible quasi-orders on X^ω can be obtained from the principal quasi-orders.

Proposition 5.1 *Let $L \subseteq X^\omega$ be an ω-language. Then:*
(i) The principal quasi-order $\sigma(L)$ is compatible, i.e., $u\ \sigma(L)\ v$ implies $xu\ \sigma(L)\ xv$ for all $x \in X^*$.
(ii) For every $w \in W(L)$ and $u \in X^\omega$ we have $w\ \sigma(L)\ u$.
(iii) If L is a quasi-separative ω-language, then $\sigma(L)$ is a compatible partial order on $X^\omega \backslash W(L)$.

Proof. *(i)* Let $u\ \sigma(L)\ v$ and $x \in X^*$. We have to show that $xu\ \sigma(L)\ xv$, that is, $L(xu)^{-1} \subseteq L(xv)^{-1}$. Suppose first that $Lu^{-1} \neq \emptyset$. If $y \in L(xu)^{-1}$, then $yxu \in L$, $yx \in Lu^{-1} \subseteq Lv^{-1}$ which shows that $yxv \in L$ and $y \in L(xv)^{-1}$. This implies $L(xu)^{-1} \subseteq L(yu)^{-1}$, i.e., $xu\ \sigma(L)\ xv$.

Suppose now that $Lu^{-1} = \emptyset$, that is, $u \in W(L)$. Since $W(L)$ is a left ω-ideal, $xu \in W(L)$ and $L(xu)^{-1} = \emptyset \subseteq L(xv)^{-1}$. Therefore $xu\ \sigma(L)\ xv$.

(ii) If $w \in W(L)$, then $Lw^{-1} = \emptyset \subseteq Lu^{-1}$, hence $w\sigma(L)u$.

(iii) By *(i)*, $\sigma(L)$ is a compatible quasi-order. Suppose $u\ \sigma(L)\ v$ and $v\ \sigma(L)\ u$ with $u, v \notin W(L)$. Then $Lu^{-1} \subseteq Lv^{-1}$ and $Lv^{-1} \subseteq Lu^{-1}$, hence $Lu^{-1} = Lv^{-1} \neq \emptyset$. Since L is quasi-separative, we have $u = v$ and therefore $\sigma(L)$ is anti-symmetric on $X^\omega \backslash W(L)$. □

Let $X = \{a, b\}$ and let X^* be listed under the lexicographic order:

$$X^* = \{a, b, a^2, ab, ba, b^2, \cdots\}$$

Let $u = aba^2abbab^2 \cdots$ be the catenation of the words from the above listing. Construct the sequence:

$$u_1 = aba^2abbab^2 \ldots,\ \ u_2 = ba^2abbab^2 \ldots,\ \ u_3 = a^2abbab^2 \ldots,\ \ \ldots$$

Let $L = \{u_1, u_2, u_3, \ldots\}$. Then

$$Lu_1^{-1} = \{1\},\ Lu_2^{-1} = \{1, a\},\ Lu_3^{-1} = \{1, a, ab\}, \ldots$$

Clearly the ω-language L is quasi-separative and $Lu_i^{-1} \subset Lu_{i+1}^{-1}$. Hence $\sigma(L)$ is a compatible partial order on $X^\omega \backslash W(L) = L$:

$$u_1\ \sigma(L)\ u_2\ \sigma(L)\ u_3\ \sigma(L) \cdots \sigma(L)\ u_i \sigma(L)\ u_{i+1}\ \sigma(L) \cdots$$

Let σ be a quasi-order on X^ω. An *upper section* is a nonempty subset S such that $u \in S$ and $u\ \sigma\ x$ implies $x \in S$. For every $u \in X^\omega$, the set $[u) = \{x \in X^\omega | u\, \sigma\, x\}$ is an upper section called the *monogenic upper section* generated by u.

Lemma 5.1 *If σ is a compatible quasi-order of X^ω and if $L = [u)$ is a monogenic upper section of σ, then $\sigma \subseteq \sigma(L)$.*

Proof. Suppose that $r\ \sigma\ s$ and let $x \in Lr^{-1}$. Then $xr \in L$ and $u\, \sigma\, xr$. Since σ is compatible, $xr\ \sigma\ xs$ and $u\ \sigma\ xs$. Hence $xs \in L$, $x \in Ls^{-1}$ and $Lr^{-1} \subseteq Ls^{-1}$. Therefore $r\ \sigma(L)\ s$ and $\sigma \subseteq \sigma(L)$. □

Proposition 5.2 *(i) If $\Lambda = \{L_i \mid i \in I\}$ is a family of ω-languages, then the relation $\sigma(\Lambda)$ defined by*

$$\sigma(\Lambda) = \bigcap_{i \in I} \sigma(L_i)$$

is a compatible quasi-order on X^ω.

(ii) If σ is a compatible quasi-order on X^ω, then there exists a family of ω-languages $\Lambda = \{L_i \mid i \in I\}$ such that $\sigma = \sigma(\Lambda)$.

Proof. *(i)* Immediate because the intersection of compatible partial orders is a compatible partial order.

(ii) Take for the family $\Lambda = \{L_i \mid i \in I\}$ the set of all monogenic upper sections L_i of σ. We will show that $\sigma = \sigma(\Lambda)$. First, by Lemma 5.1, we have $\sigma \subseteq \bigcap_{i \in I} \sigma(L_i) = \sigma(\Lambda)$. Suppose that $\sigma \neq \sigma(\Lambda)$. Then there exist $r, s \in X^\omega$ such that $r\sigma(\Lambda)s$ and r not in relation σ with s. If $K = \{x \in X^* \mid r\sigma\, x\}$ is the monogenic section generated by r, then $r \in K$ and $1 \in Kr^{-1}$. Since $K \in \Lambda$, then $r\ \sigma(K)\ s$ and $Kr^{-1} \subseteq Ks^{-1}$. Consequently, $1 \in Ks^{-1}$, $s \in K$ and $r\ \sigma\ s$, a contradiction. Therefore $\sigma = \sigma(\Lambda)$. □

A family $\Lambda = \{L_i \mid i \in I\}$ of ω-languages $L_i \subseteq X^\omega$ is said to be *strong if*

$$L_i u^{-1} = L_i v^{-1}\ \ \forall i \in I \ \ \Rightarrow \ \ u = v$$

Proposition 5.3 *(i) If $\Lambda = \{L_i \mid i \in I\}$ is a strong family of ω-languages, then the relation $\sigma(\Lambda)$ defined by*

$$\sigma(\Lambda) = \bigcap_{i \in I} \sigma(L_i)$$

is a compatible partial order on X^ω.

(ii) If σ is a compatible partial order on X^ω, then there exists a strong family of ω-languages $\Lambda = \{L_i \mid i \in I\}$ such that $\sigma = \sigma(\Lambda)$.

Proof. *(i)* By Proposition 5.2, $\sigma(\Lambda)$ is a compatible quasi-order. Since the family Λ is strong, this implies that the quasi-order $\sigma(\Lambda)$ is antisymmetic and hence a compatible partial order.

(ii) As in the proof of the preceding proposition, take for the family $\Lambda = \{L_i \mid i \in I\}$ the set of all monogenic upper sections L_i of σ. Then we have $\sigma = \sigma(\Lambda)$. What is left to show is that the family Λ is strong.

Suppose that Λ is not strong. Then there exist $u, v \in X^\omega$, $u \neq v$, such that $L_i u^{-1} = L_i v^{-1}$ for all the monogenic upper sections L_i of σ. Let $U = [u)$ and $V = [v)$ be the monogenic sections of u, respectively v. Since $1 \in Uu^{-1}$, $1 \in Uv^{-1}$, $v = 1.v \in U$ and $u\ \sigma\ v$. Using the same argument, it can be shown that $v\ \sigma\ u$. Since σ is a partial order, this implies $u = v$, a contradiction. □

If L is a separative ω-language, then the relation $\sigma(L)$ defined by $u\ \sigma(L)\ v \Leftrightarrow Lu^{-1} \subseteq Lv^{-1}$ is an order relation on X^ω that will be called the *principal order* associated with L. It is easy to see that this order is the identity iff $Lu^{-1} \subseteq Lv^{-1}$ implies $u = v$.

Proposition 5.4 *Let $L \subseteq X^w$ be a separative ω-language. Then the principlal order $\sigma(L)$ is compatible.*

Proof. Let $u\ \sigma(L)\ v$ and $x \in X^*$. Since L is separative, then L is ω-dense and hence $L(xu)^{-1} \neq \emptyset$. Let $y \in L(xu)^{-1}$. This implies $yux \in L$, $yx \subseteq Lu^{-1} = Lv^{-1}$, $yxv \in L$ and $y \in L(xv)^{-1}$. Therefore $L(xu)^{-1} \subseteq L(xv)^{-1}$. Similarly $L(xv)^{-1} \subseteq L(xu)^{-1}$. Hence $L(xu)^{-1} = L(xv)^{-1}$ and $xu\ \sigma(L)\ xv$. □

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